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Liquid Crystals

Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713926090

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Online publication date: 06 August 2010

To cite this Article Blake, George I. and Leslie, Frank M.(1998) 'A backflow effect in smectic C liquid crystals', Liquid Crystals, 25: 3, 319 — 327

To link to this Article: DOI: 10.1080/026782998206119

URL: http://dx.doi.org/10.1080/026782998206119

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LIQUID CRYSTALS, 1998, VOL. 25, NO. 3, 319–327

A backflow effect in smectic C liquid crystals

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(*Received 30 October 1997; accepted 29 November 1997*)

This paper discusses the influence upon a smectic C liquid crystal cell of backflow induced by the relaxation of alignment following the removal of a strong electric or magnetic field. Our study, based upon a recently proposed continuum theory, concentrates upon the homeotropic configuration in which the smectic layers are parallel to the boundary plates, but some consideration is also given to the bookshelf geometry. Although the governing equations prove to be rather complex, some progress is possible analytically by repeating an approximation made in the corresponding problem for a nematic.

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1. Introduction

A problem in nematic liquid crystals that gave rise to a good deal of theoretical and experimental interest was that of 'optical bounce' in a twisted nematic cell, this demonstrating the relevance of flow effects to display applications. Gerritsma *et al.* [1] observed that upon the removal of a strong electric field applied across a twisted nematic cell, the optical transmission does not decrease monotonically to zero as one might expect; instead, following an initial decrease, it increases to a value slightly less than the initial level before finally decaying to zero. Van Doorn [2] explains this phenomenon as a consequence of flow induced close to the cell walls, commonly called 'backflow', by the rapidly relaxing alignment following the removal of the field. This flow essentially creates a shear flow in the centre of the cell which causes a temporary reversal of the relaxation in that region, before the alignment finally relaxes as one anticipates. His explanation is borne out by subsequent theoretical studies [3–5].

With the possibility of fast, bistable ferroelectric liquid crystal display devices as discussed by Clark and Lagerwall [6], attention has largely turned in the last decade or so from nematic to smectic liquid crystals. Clearly it is reasonable to question whether similar flow effects can also occur in this more complex type of liquid crystal. Consequently this paper employs the continuum theory for smectics proposed recently by Leslie *et al.* [7] to investigate the possibility of backflow influencing relaxation of alignment in smectic C liquid crystals when a magnetic or electric field is removed. Not surprisingly, our attention is mostly given to the somewhat simpler homeotropic configuration in which the layering is parallel to the plates. Although the continuum equations for smectics are more complex than those for nematics, it does prove possible to reproduce to a large extent the calculations of Clark and Leslie [5] for nematic liquid crystals, the only significant difference being the need to include two velocity components rather than one.

In summary, the following section of the paper gives a brief account of the continuum theory of Leslie *et al.* for smectic C liquid crystals. Thereafter we consider a situation for a homeotropically aligned sample which is analogous to the optical bounce problem in nematics, discussing two approximate analytical solutions. The first approximates behaviour close to a boundary surface, and largely motivates the second more practical solution. The final section briefly discusses the effect of backflow in the bookshelf geometry.

2. Continuum theory

The continuum theory of Leslie *et al.* [7] for smectic C liquid crystals invokes a number of simplifying assumptions, which help to avoid undue mathematical complexity, but of course limit its range of application. In particular it assumes that the smectic layer spacing remains constant (although the layers can deform) and also that the tilt of alignment with respect to the layer normal remains fixed. The former assumption certainly appears reasonable in many situations, as does the latter, although excluding pretransitional effects. Given these constraints it is possible to describe smectic configurations using two orthonormal vectors, one the layer

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unit normal $\mathbf{a}$ and the second a unit vector $\mathbf{c}$ giving the direction of tilt with respect to the layer normal (cf. de Gennes and Prost [8]) so that

$$\mathbf{a}\cdot\mathbf{a}=\mathbf{c}\cdot\mathbf{c}=1, \qquad \mathbf{a}\cdot\mathbf{c}=0 \tag{1}$$

and in the absence of defects in the layering one must add

$$\operatorname{curl}\mathbf{a}=0 \tag{2}$$

as Oseen [9], and de Gennes and Prost [8] discuss. Moreover the theory assumes isothermal conditions, and also not unreasonably incompressibility, with the result that the velocity vector $\mathbf{v}$ is subject to the constraint

$$\operatorname{div}\mathbf{v}=0 \tag{3}$$

and the density ρ is constant.

The balance laws are those of classical continuum mechanics, namely balance of linear and angular momentum except that the latter includes terms commonly omitted. In Cartesian tensor notation the former takes the form

$$\rho\dot{v}_i=\rho F_i+t_{ij,j} \tag{4}$$

$\mathbf{F}$ denoting body force per unit mass, $\mathbf{t}$ the stress tensor, and the superposed dot the material time derivative, while the latter reduces to

$$0=\rho K_i+e_{ijk}t_{kj}+l_{ij,j} \tag{5}$$

with $\mathbf{K}$ denoting the external body moment per unit mass and $\mathbf{l}$ the couple stress tensor. In the above a repeated index is subject to the summation convention, a comma preceding a suffix denotes a partial derivative with respect to the corresponding spatial coordinate, and e_{ijk} is the alternator.

The constitutive relations for stress and couple stress are

$$t_{ij}=-p\delta_{ij}+\beta_p e_{pjk}a_{k,i}-\frac{\partial W}{\partial a_{k,j}}a_{k,i}-\frac{\partial W}{\partial c_{k,j}}c_{k,i}+\tilde{t}_{ij} \tag{6}$$

$$l_{ij}=\beta_p a_p\delta_{ij}-\beta_i a_j+e_{ipq}\left(a_p\frac{\partial W}{\partial a_{q,j}}+c_p\frac{\partial W}{\partial c_{q,j}}\right) \tag{7}$$

where the pressure p and the vector β arise on account of the constraints (3) and (2), respectively. The energy W can be expressed in several equivalent forms [10] and here we write

$$\begin{aligned}2W=&K_1(a_{i,i})^2+(K_2-K_4)(c_{i,i})^2+(K_3-K_4)c_{i,p}c_pc_{i,q}c_q\\&+K_4c_{i,j}c_{i,j}+(K_5-K_3)(c_ia_{i,j}c_j)^2\\&+2K_6a_{i,i}(c_ja_{j,k}c_k)-2K_7c_{i,p}c_pc_{i,q}a_q\\&+2(K_8-K_7)c_{i,i}(c_ja_{j,k}c_k)+2K_9a_{i,i}c_{j,j}\end{aligned} \tag{8}$$

the Ks constants. Following an argument proposed originally by Ericksen [11] it follows that

$$K_2>0, \qquad K_3>0, \qquad K_4>0 \tag{9}$$

together with other inequalities not required in this paper. The viscous stress tensor $\tilde{\mathbf{t}}$ is the sum of a symmetric part

$$\begin{aligned}(\tilde{t}_{ij})=&\mu_0D_{ij}+\mu_1a_kD_{kp}a_pa_ia_j+\mu_2(D_{ik}a_ka_j+D_{jk}a_ka_i)\\&+\mu_3c_kD_{kp}c_pc_ic_j+\mu_4(D_{ik}c_kc_j+D_{jk}c_kc_i)\\&+\mu_5(a_ic_j+a_jc_i)c_kD_{kp}a_p+\lambda_1(A_ia_j+A_ja_i)\\&+\lambda_2(C_ic_j+C_jc_i)+\lambda_3(a_ic_j+a_jc_i)A_kc_k\\&+\kappa_1(D_{ik}a_kc_j+D_{jk}a_kc_i+D_{ik}c_ka_j+D_{jk}c_ka_i)\\&+\kappa_2[(a_ic_j+a_jc_i)a_kD_{kp}a_p+2a_kD_{kp}c_pa_ia_j]\\&+\kappa_3[(a_ic_j+a_jc_i)c_kD_{kp}c_p+2a_kD_{kp}c_pc_ic_j]\\&+\tau_1(C_ia_j+C_ja_i)+\tau_2(A_ic_j+A_jc_i)\\&+2\tau_3A_kc_ka_ia_j+2\tau_4A_kc_kc_ic_j\end{aligned} \tag{10}$$

and a skew-symmetric part

$$\begin{aligned}[\tilde{t}_{ij}]=&\lambda_1(D_{jk}a_ka_i-D_{ik}a_ka_j)+\lambda_2(D_{jk}c_kc_i-D_{ik}c_kc_j)\\&+\lambda_3(a_ic_j-a_jc_i)c_kD_{kp}a_p+\lambda_4(A_ja_i-A_ia_j)\\&+\lambda_5(C_jc_i-C_ic_j)+\lambda_6(a_ic_j-a_jc_i)A_kc_k\\&+\tau_1(D_{jk}a_kc_i-D_{ik}a_kc_j)+\tau_2(D_{jk}c_ka_i-D_{ik}c_ka_j)\\&+\tau_3(a_ic_j-a_jc_i)a_kD_{kp}a_p+\tau_4(a_ic_j-a_jc_i)c_kD_{kp}c_p\\&+\tau_5(A_jc_i-A_ic_j+C_ja_i-C_ia_j)\end{aligned} \tag{11}$$

where

$$\begin{aligned}2D_{ij}=v_{i,j}+v_{j,i}, \qquad 2W_{ij}=v_{i,j}-v_{j,i}\\A_i=\dot{a}_i-W_{ik}a_k, \qquad C_i=\dot{c}_i-W_{ik}c_k\end{aligned} \tag{12}$$

and the coefficients are constants.

The intrinsic viscous moment in equation (5) can be expressed as

$$e_{ijk}[\tilde{t}_{kj}]=e_{ijk}(a_j\tilde{g}^a_k+c_j\tilde{g}^c_k) \tag{13}$$

where

$$\begin{aligned}\tilde{g}^a_i=&-2(\lambda_1D_{ik}a_k+\lambda_3a_kD_{kp}c_pc_i+\lambda_4A_i+\lambda_6A_kc_kc_i\\&+\tau_2D_{ik}c_k+\tau_3a_pD_{pk}a_kc_i+\tau_4c_pD_{pk}c_kc_i+\tau_5C_i)\end{aligned} \tag{14}$$

$$\tilde{g}^c_i=-2(\lambda_2D_{ik}c_k+\lambda_5C_i+\tau_1D_{ik}a_k+\tau_5A_i). \tag{15}$$

Likewise the external body moment arising from external magnetic or electric fields can be written

$$\rho K_i=e_{ijk}(a_jG^a_k+c_jG^c_k). \tag{16}$$

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As a consequence of equations (13) and (16) it is possible to rewrite the angular momentum (5) as

$$\left(\frac{\partial W}{\partial a_{i,j}}\right)_{,j} - \frac{\partial W}{\partial a_i} + G_i^a + \tilde{g}_i^a + \gamma a_i + \mu c_i + e_{ijk}\beta_{k,j} = 0 \tag{17}$$

$$\left(\frac{\partial W}{\partial c_{i,j}}\right)_{,j} - \frac{\partial W}{\partial c_i} + G_i^c + \tilde{g}_i^c + \tau c_i + \mu a_i = 0 \tag{18}$$

the scalars γ, μ and τ being Lagrangian multipliers arising from the constraints (1). Also the balance of linear momentum can be written as

$$\rho \dot{v}_i = \rho F_i - \tilde{p}_{,i} + \tilde{g}_k^a a_{k,i} + \tilde{g}_k^c c_{k,i} + \tilde{t}_{ij,j} \tag{19}$$

with

$$\tilde{p} = p + W - \psi \tag{20}$$

ψ denoting the energy associated with the external magnetic or electric field. The forms for ψ, $\mathbf{G}^a$ and $\mathbf{G}^c$ are not given since they are not required in what follows. Finally the viscous dissipation inequality reduces to

$$(\tilde{t}_{ij}) D_{ij} - \tilde{g}_i^a A_i - \tilde{g}_i^c C_i \geqslant 0 \tag{21}$$

which is required to place restrictions upon viscous coefficients in our analysis.

3. Backflow in the homeotropic geometry

Consider a uniformly aligned smectic C liquid crystal in the homeotropic geometry confined between two parallel, flat plates a distance $2d$ apart, so that the layers are everywhere parallel to the plates. With an appropriate choice of Cartesian axes, the layers and the plates lie in x,y-planes and the alignment in x,z-planes. Assuming strong anchoring at the plates, application of a sufficiently strong magnetic field in the direction of the y-axis leads to a distortion of the **c**-director, provided that the liquid crystal has a positive diamagnetic anisotropy. Consequently, if the layers remain undistorted by the application of such a field, the directors **a** and **c** take the forms

$$\mathbf{a} = (0, 0, 1), \qquad \mathbf{c} = (\cos\phi, \sin\phi, 0) \tag{22}$$

where ϕ is the angle that **c** makes with the x-axis. If the distortion of **c** is uniform in the x and y directions, ϕ is solely a function of z and t. Below we examine the relaxation of the alignment following the removal of the magnetic field.

An inspection of equations (17)–(19) suggests that the problem is likely to be over-defined if one simply examines solutions of the form (22), and indeed it is necessary to introduce two components of velocity for the system to be well defined, instead of just one as in the corresponding problem for nematics. Therefore one must also consider a velocity field of the form

$$\mathbf{v} = (u, v, 0) \tag{23}$$

where u and v depend only upon z and t. Our choice of solution is entirely consistent with the constraints (1)–(3), and not unreasonably the pressure p, the vector β and the Lagrange multipliers γ, μ and τ are also assumed to be functions of only z and t. With the forms of solution (22) and (23), equations (17)–(19) yield

$$\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial z}\left[(\eta_1 + \eta_2\cos^2\phi)\frac{\partial u}{\partial z} + \eta_2 \sin\phi\cos\phi\frac{\partial v}{\partial z} + (\tau_5 - \tau_1)\sin\phi\frac{\partial \phi}{\partial t}\right] \tag{24}$$

$$\rho\frac{\partial v}{\partial t} = \frac{\partial}{\partial z}\left[\eta_2 \sin\phi\cos\phi\frac{\partial u}{\partial z} + (\eta_1 + \eta_2\sin^2\phi)\frac{\partial v}{\partial z} - (\tau_5 - \tau_1)\cos\phi\frac{\partial \phi}{\partial t}\right] \tag{25}$$

$$K_4\frac{\partial^2\phi}{\partial z^2} - 2\lambda_5\frac{\partial\phi}{\partial t} + (\tau_5 - \tau_1)\left(\cos\phi\frac{\partial v}{\partial z} - \sin\phi\frac{\partial u}{\partial z}\right) = 0 \tag{26}$$

along with expressions for the pressure p, the vector β and the multipliers γ, μ and τ, which are not required. In equations (24) and (25) the viscosities η_1 and η_2 are given by

$$\begin{aligned} 2\eta_1 &= \mu_0 + \mu_2 - 2\lambda_1 + \lambda_4 \\ 2\eta_2 &= \mu_4 + \mu_5 + 2\lambda_2 - 2\lambda_3 + \lambda_5 + \lambda_6. \end{aligned} \tag{27}$$

With appropriate choices for the velocity and director fields, one can show using the inequality (21) that

$$\eta_1 > 0, \qquad \eta_1 + \eta_2 > 0, \qquad \lambda_5 > 0. \tag{28}$$

By assuming strong anchoring of the alignment and no-slip of the velocity, the boundary and initial conditions read

$$\phi(\pm d, t) = u(\pm d, t) = v(\pm d, t) = 0, \qquad t \geqslant 0 \tag{29}$$

$$\phi(z, 0) = \phi_0(z), \qquad u(z, 0) = v(z, 0) = 0, \qquad -d < z < d \tag{30}$$

where ϕ_0 is a known even function of z, having taken our origin of coordinates midway between the plates.

The system of equations (24)–(26) is rather complicated and it appears that a numerical approach is the only option available. However, their form is similar to the equations for the corresponding backflow problem in nematics [5], and by making certain approximations Clark and Leslie were able to make some analytical progress. By analogy we introduce similar simplifying

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assumptions which allow us to continue without recourse to numerical techniques.

3.1. *The half-space problem*

To study the behaviour of the **c**-director and the induced flow in the vicinity of a boundary, consider a half-space entirely filled with a uniformly aligned smectic C liquid crystal with its layers parallel to the bounding surface. As above, it is convenient to choose a Cartesian coordinate system for which the **c**-director is initially parallel to the x-axis and the x,y-plane is coincident with the boundary, and again assume that there is strong anchoring at the boundary. If a sufficiently strong magnetic field is applied parallel to the y-axis, that is in the plane of the layers but perpendicular to the initial alignment of the **c**-director, one expects **c** to become everywhere approximately parallel to the direction of the applied field, except in a narrow region close to the boundary. Therefore a reasonable approximation to the initial condition on the azimuthal angle ϕ in this case is

$$\phi(z,0)=\pi/2, \qquad z>0. \tag{31}$$

Once **c** is fully re-orientated, the field is removed.

To simplify the governing equations (24)–(26) it seems reasonable to assume that during the first stages of re-orientation good approximations to solutions are given by replacing the variable coefficients by their values initially. In this event, replacing ϕ in these coefficients by the value $\pi/2$ one obtains

$$\begin{aligned}\rho\frac{\partial u}{\partial t}&=\eta_1\frac{\partial^2 u}{\partial z^2}+(\tau_5-\tau_1)\frac{\partial^2\phi}{\partial z\partial t}\\ \rho\frac{\partial v}{\partial t}&=(\eta_1+\eta_2)\frac{\partial^2 v}{\partial z^2}\\ 2\lambda_5\frac{\partial\phi}{\partial t}&=K_4\frac{\partial^2\phi}{\partial z^2}-(\tau_5-\tau_1)\frac{\partial u}{\partial z}.\end{aligned} \tag{32}$$

As well as strong anchoring we assume no-slip at the boundary, and thus our boundary conditions are

$$\phi(0,t)=0, \qquad u(0,t)=v(0,t)=0, \qquad t\geqslant 0. \tag{33}$$

Also, it seems reasonable to assume that the induced flow decays with distance from the wall, and therefore we add

$$u(z,t)\to 0, \qquad v(z,t)\to 0 \quad \text{as } z\to\infty \tag{34}$$

as well as the initial conditions

$$u(z,0)=0, \qquad v(z,0)=0, \qquad z>0. \tag{35}$$

Immediately the second of equations (32) subject to the above conditions on the component v yields

$$v(z,t)=0, \qquad z,t>0 \tag{36}$$

so that this component of flow plays no role in the initial relaxation.

The form of the coupled pair of partial differential equations remaining in equations (32) plus the boundary and initial conditions (31), (33)–(35) suggest the use of a similarity variable, and straightforward dimensional analysis leads one to

$$\begin{aligned}\phi&=F(\zeta), & u&=(k/t)^{1/2}G(\zeta)\\ \zeta&=z/2(kt)^{1/2}, & k&=K_4/2\lambda_5.\end{aligned} \tag{37}$$

As a consequence the remaining equations (32) become

$$\lambda_5(F''+2\zeta F')=(\tau_5-\tau_1)G' \tag{38}$$

$$\eta_1 G''+\eta_1\varepsilon(\zeta G'+G)+(\tau_1-\tau_5)(\zeta F''+F')=0 \tag{39}$$

with

$$\varepsilon=\rho K_4/\eta_1\lambda_5 \tag{40}$$

and the prime denoting differentiation with respect to ζ. Also the boundary and initial conditions reduce to

$$F(0)=0, \qquad F(\zeta)\to\pi/2 \quad \text{as } \zeta\to\infty \tag{41}$$

$$G(0)=0, \qquad \zeta G(\zeta)\to 0 \quad \text{as } \zeta\to\infty. \tag{42}$$

The parameter ε is positive on account of the inequalities (9) and (28).

Equation (39) readily integrates to yield

$$\eta_1 G'+\eta_1\varepsilon\zeta G=(\tau_5-\tau_1)\zeta F' \tag{43}$$

the constant of integration being zero on account of conditions at large ζ. This integral allows one to eliminate F from equation (38) to obtain an equation for G which is readily solved by the change of independent variable from ζ to ζ^2. Having found G, F follows quickly from equation (43). In this way one readily obtains

$$\begin{aligned}F(\zeta)&=\frac{\pi}{2\Delta}\left[(2\nu_1-\varepsilon)\operatorname{erf}(\nu_1^{1/2}\zeta)-\left(\frac{\nu_1}{\nu_2}\right)^{1/2}(2\nu_2-\varepsilon)\operatorname{erf}(\nu_2^{1/2}\zeta)\right]\\ G(\zeta)&=\frac{(\tau_1-\tau_5)(\pi\nu_1)^{1/2}}{\eta_1\Delta}\left[\exp(-\nu_1\zeta^2)-\exp(-\nu_2\zeta^2)\right]\end{aligned} \tag{44}$$

where

$$\begin{aligned}\Delta&=(2\nu_1-\varepsilon)-(2\nu_2-\varepsilon)\left(\frac{\nu_1}{\nu_2}\right)^{1/2}\\ \operatorname{erf}(x)&=\frac{2}{\pi^{1/2}}\int_0^x\exp(-s^2)\,\mathrm{d}s\\ 4\nu_1&=2\mu+\varepsilon+\left[(2\mu+\varepsilon)^2-8\varepsilon\right]^{1/2}\\ 4\nu_2&=2\mu+\varepsilon-\left[(2\mu+\varepsilon)^2-8\varepsilon\right]^{1/2}.\end{aligned} \tag{45}$$

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The parameter μ is given by

$$\mu = 1 - \frac{(\tau_5 - \tau_1)^2}{2\eta_1 \lambda_5} \tag{46}$$

and can be shown to be positive using the inequality (21).

Since the viscous coefficients of a smectic C liquid crystal are invariably much larger than the elastic constants, one has

$$\mu \gg \varepsilon \tag{47}$$

and ν_1 and ν_2 are clearly positive. To the first order in ε they are given by

$$\nu_1 = \mu, \qquad \nu_2 = \varepsilon/2\mu \tag{48}$$

and therefore the expressions (44) simplify considerably to give

$$\begin{aligned} \phi &\simeq \frac{\pi}{2}\,\mathrm{erf}[(\mu)^{1/2}\zeta] \\ u &\simeq \frac{(\tau_1 - \tau_5)}{2\eta_1}\left(\frac{\pi k}{\mu t}\right)^{1/2}[\exp(-\mu\zeta^2) - \exp(-\varepsilon\zeta^2/2\mu)] \end{aligned} \tag{49}$$

which are more useful in practice

The plots for the angle ϕ and the flow component u against time at fixed distances from the wall are shown in figures 1 and 2. The values taken for the various material parameters are

$$\begin{aligned} &\tau_5 - \tau_1 = 2{\cdot}4\ \mathrm{Kg\,m^{-1}\,s^{-1}}, \qquad \eta_1 = 1{\cdot}9\ \mathrm{Kg\,m^{-1}\,s^{-1}} \\ &\lambda_5 = 3{\cdot}0\ \mathrm{Kg\,m^{-1}\,s^{-1}}, \qquad K_4 = 10^{-11}\ \mathrm{N} \\ &\rho = 1\ \mathrm{Kg\,m^{-3}} \end{aligned} \tag{50}$$

all in SI units. The values for the viscosities are those obtained by Leslie and Gill [12] from earlier light-scattering data, while that for the elastic constant is simply a typical nematic value. The graph for ϕ yields no surprises in that the angle relaxes straightforwardly to zero after an initial delay dependent upon distance from the wall, the further from the boundary the longer is the time for **c** to return to its position before the field was applied. However, the graph for the flow component against time is more interesting. It illustrates that soon after the field is removed the induced flow is quite large at distances of one or two microns from the boundary. Since most cells used in the laboratory or devices have gap widths of the order of microns, this flow is obviously significant. Further, since $\tau_5 - \tau_1$ is expected to be positive in most smectic C materials [13], the latter of equations (49) implies that the induced flow is in the direction of the positive x-axis. Following the arguments of Gerritsma *et al.* [1], and Clark and Leslie [5] the above analysis implies that kickback is rather likely in a homeotropically aligned smectic C material. These conclusions therefore motivate a more realistic analysis of this problem.

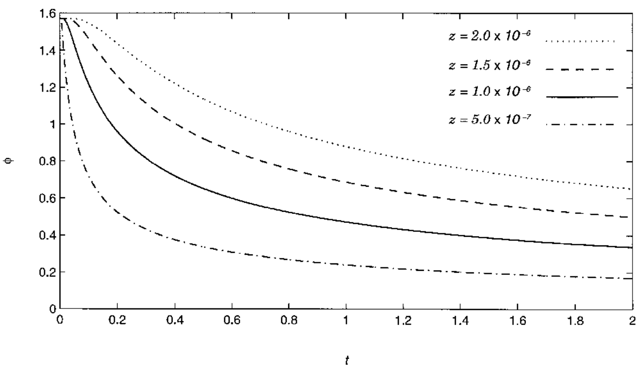

Figure 1. Plot of the relaxation of ϕ at various distances from the boundary.

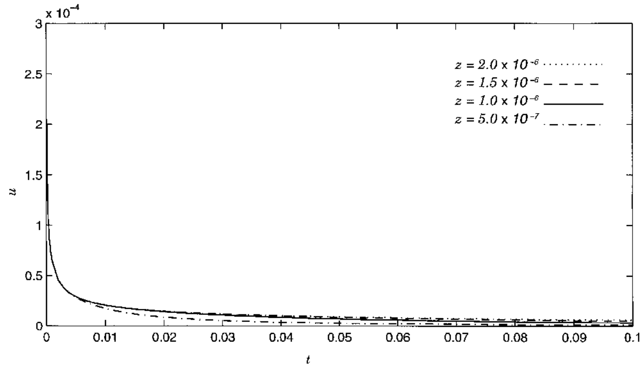

Figure 2. Plot of the velocity, u, at various distances from the boundary.

3.2. *The cell problem*

The main obstacle towards solving analytically the system of equations (24)–(26) subject to the conditions (29) and (30) is the fact that most of the coefficients depend upon the angle ϕ making the system non-linear. Some progress with this difficult problem is possible by following the approach by Clark and Leslie [5] and setting these coefficients equal to constants. In this event the equations become

$$\left.\begin{aligned} \rho\frac{\partial u}{\partial t} &= m\frac{\partial^2 u}{\partial z^2} + p\frac{\partial^2 v}{\partial z^2} + a\frac{\partial^2 \phi}{\partial z \partial t} \\ \rho\frac{\partial v}{\partial t} &= p\frac{\partial^2 u}{\partial z^2} + n\frac{\partial^2 v}{\partial z^2} + b\frac{\partial^2 \phi}{\partial z \partial t} \\ 2\lambda_5\frac{\partial \phi}{\partial t} &= K_4\frac{\partial^2 \phi}{\partial z^2} - a\frac{\partial u}{\partial z} - b\frac{\partial v}{\partial z} \end{aligned}\right\} \tag{51}$$

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where

$$a = (\tau_5 - \tau_1)\sin\hat{\phi}, \qquad b = (\tau_1 - \tau_5)\cos\hat{\phi}$$
$$m = \eta_1 + \eta_2\cos^2\hat{\phi}, \qquad n = \eta_1 + \eta_2\sin^2\hat{\phi} \tag{52}$$
$$p = \eta_2\sin\hat{\phi}\cos\hat{\phi}$$

the circumflex denoting the fact that the value of ϕ is fixed. These equations are subject to the boundary and initial conditions (29) and (30).

Introducing new non-dimensional variables

$$\zeta = \frac{z}{d}, \qquad \tau = \frac{kt}{d^2}, \qquad \tilde{u} = \frac{du}{k}, \qquad \tilde{v} = \frac{dv}{k}, \qquad k = \frac{K_4}{2\lambda_5} \tag{53}$$

the equations (51) become

$$\left.\begin{aligned}
\varepsilon\frac{\partial\tilde{u}}{\partial\tau} &= \frac{\partial^2\tilde{u}}{\partial\zeta^2} + \alpha\frac{\partial^2\tilde{v}}{\partial\zeta^2} + \gamma\frac{\partial^2\phi}{\partial\zeta\partial\tau}\\
\varepsilon\frac{\partial\tilde{v}}{\partial\tau} &= \frac{\partial^2\tilde{v}}{\partial\zeta^2} + \beta\frac{\partial^2\tilde{u}}{\partial\zeta^2} + \delta\frac{\partial^2\phi}{\partial\zeta\partial\tau}\\
\frac{\partial\phi}{\partial\tau} &= \frac{\partial^2\phi}{\partial\zeta^2} - \omega\frac{\partial\tilde{u}}{\partial\zeta} - \kappa\frac{\partial\tilde{v}}{\partial\zeta}
\end{aligned}\right\} \tag{54}$$

where

$$\varepsilon = \frac{\rho k}{m}, \qquad \alpha = \frac{p}{m}, \qquad \beta = \frac{p}{n}, \qquad \gamma = \frac{a}{m}$$
$$\delta = \frac{b}{n}, \qquad \omega = \frac{a}{2\lambda_5}, \qquad \kappa = \frac{b}{2\lambda_5}. \tag{55}$$

Also the boundary and initial conditions now take the form

$$\phi(\pm 1, \tau) = \tilde{u}(\pm 1, \tau) = \tilde{v}(\pm 1, \tau) = 0, \qquad \tau > 0$$
$$\phi(\zeta, 0) = \phi_0(\zeta), \qquad \tilde{u}(\zeta, 0) = \tilde{v}(\zeta, 0) = 0, \qquad -1 < \zeta < 1. \tag{56}$$

As remarked above, the coefficient ε is very small compared with unity, and therefore we neglect the associated inertial terms in the first two of equations (54), with the implication that the initial conditions on the flow can no longer be satisfied. However, as a consequence the first two of equations (54) yield

$$\begin{aligned}
(1-\alpha\beta)\frac{\partial^2\tilde{u}}{\partial\zeta^2} &= (\alpha\delta - \gamma)\frac{\partial^2\phi}{\partial\zeta\partial\tau}\\
(1-\alpha\beta)\frac{\partial^2\tilde{v}}{\partial\zeta^2} &= (\beta\gamma - \delta)\frac{\partial^2\phi}{\partial\zeta\partial\tau}
\end{aligned} \tag{57}$$

and following a differentiation with respect to ζ, the remaining equation reduces to

$$\frac{\partial^3\phi}{\partial\zeta^3} = \mu\frac{\partial^2\phi}{\partial\zeta\partial\tau}, \qquad \mu = 1 - \frac{(\tau_5 - \tau_1)^2}{2\eta_1\lambda_5} \tag{58}$$

the coefficient μ exactly as above taking a value between zero and unity.

Upon applying the relevant boundary conditions (56), one finds by separation of variables that an appropriate solution of equation (58) is

$$\phi = A(\cos q\zeta - \cos q)\exp(-q^2\tau/\mu) \tag{59}$$

A and q being constants to be determined. It then follows from equations (57) and the boundary conditions (56) that

$$\begin{aligned}
\tilde{u} &= \frac{A(\gamma - \alpha\delta)}{(1-\alpha\beta)\mu}q(\sin q\zeta - \zeta\sin q)\exp(-q^2\tau/\mu)\\
\tilde{v} &= \frac{A(\delta - \beta\gamma)}{(1-\alpha\beta)\mu}q(\sin q\zeta - \zeta\sin q)\exp(-q^2\tau/\mu)
\end{aligned} \tag{60}$$

and for a non-trivial solution the last of equations (54) requires that

$$(1-\mu)\tan q = q. \tag{61}$$

Since this equation has infinitely many solutions for q, it follows by superposition that

$$\left.\begin{aligned}
\phi &= \sum_{n=1}^{\infty} A_n(\cos q_n\zeta - \cos q_n)\exp(-q_n^2\tau/\mu)\\
\tilde{u} &= \frac{(\gamma - \alpha\delta)}{(1-\alpha\beta)\mu}\sum_{n=1}^{\infty} A_n q_n(\sin q_n\zeta - \zeta\sin q_n)\exp(-q_n^2\tau/\mu)\\
\tilde{v} &= \frac{(\delta - \beta\gamma)}{(1-\alpha\beta)\mu}\sum_{n=1}^{\infty} A_n q_n(\sin q_n\zeta - \zeta\sin q_n)\exp(-q_n^2\tau/\mu)
\end{aligned}\right\} \tag{62}$$

where the q_n are positive roots of the equation (61). The initial condition (56) on ϕ implies that

$$\sum_{n=1}^{\infty} A_n(\cos q_n\zeta - \cos q_n) = \phi_0(\zeta), \qquad -1 < \zeta < 1 \tag{63}$$

from which, noting that the functions $\cos q_n\zeta - \cos q_n$ and $\cos q_n\zeta$ are reciprocal functions with respect to integration from -1 to 1, one obtains the coefficients A_n in the form

$$A_n = \frac{2q_n}{(q_n - \sin q_n\cos q_n)}\int_0^1 \phi_0(\zeta)\cos q_n\zeta\,\mathrm{d}\zeta. \tag{64}$$

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Further, following the removal of a very strong field, it is reasonable to select

$$\phi_0(\zeta) = \frac{\pi}{2}, \qquad -1 < \zeta < 1 \tag{65}$$

and in this case

$$A_n = \pi \cos q_n / (\sin^2 q_n - \mu) \tag{66}$$

which are more readily calculated.

Figures 3–5 provide plots for ϕ, $\tilde{u}$ and $\tilde{v}$ against reduced time τ at various distances from the cell wall. These illustrate behaviour following the removal of a very strong field, and therefore employ the approximate formulae (66). The curves were calculated using two thousand terms in the series solutions. The first 20 roots of equation (61) were found using the Rule of False Position, but thereafter the remainder were approximated by $(2n-1)\pi/2$. In our calculations the viscosity functions α, β, γ and δ were updated at each step, the angle ϕ being replaced by a more appropriate choice. The values chosen for the material parameters are our earlier choice (50), with the value for η_2 taken to be $-0{\cdot}2$ in SI units [12]. Figure 3 shows that ϕ does indeed

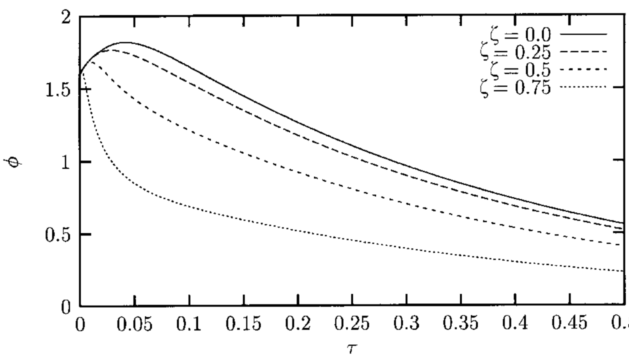

Figure 3. Plot of the relaxation of ϕ at various points in the cell.

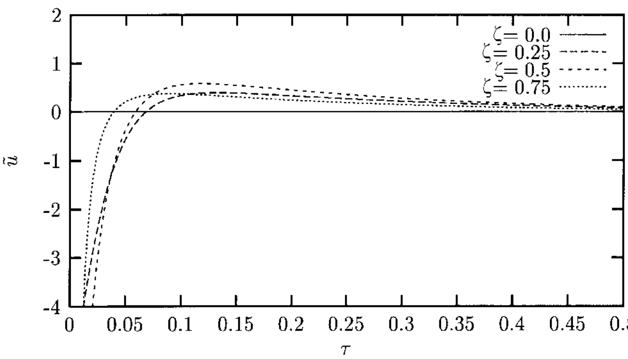

Figure 4. Plot of the non-dimensionalized x-component of velocity, $\tilde{u}$, at various points in the cell.

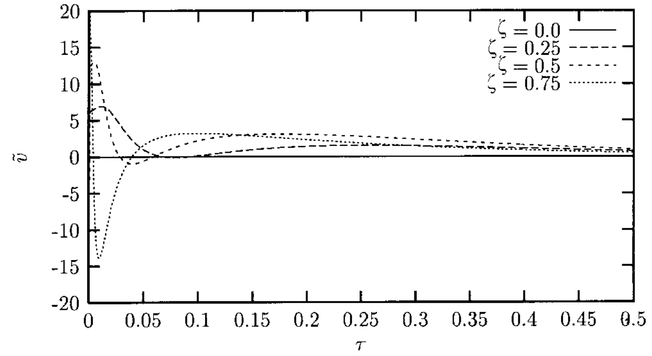

Figure 5. Plot of the non-dimensionalized y-component of velocity, $\tilde{v}$, at various points in the cell.

increase shortly after the removal of the field before decaying to zero, the effect most marked in the centre of the cell. Since our analysis neglects the fluid inertia, the initial values for the induced flow are non-zero in both figures 4 and 5.

4. Backflow in the bookshelf geometry

In this section we consider briefly similar backflow effects in a smectic C liquid crystal when the layers are everywhere perpendicular to the two parallel bounding plates. With a choice of coordinate axes such that the x,y-planes are parallel to the plates, we assume that initially the anisotropic axis lies in the x,y-plane and the **c**-director lies parallel to the y-axis. Provided that the surface anchoring of the alignment is strong and the liquid crystal has a positive diamagnetic anisotropy, application of a strong magnetic field perpendicular to the plates leads to a distortion of the uniform **c**-director profile. As above, our intention is to examine behaviour following the removal of the field by seeking solutions of equations (17)–(19) of the form

$$\mathbf{a} = (1, 0, 0), \qquad \mathbf{c} = (0, \cos\phi, \sin\phi), \qquad \mathbf{v} = (u, v, 0) \tag{67}$$

where ϕ, u and v are functions solely of z and t.

The equations for angular momentum reduce to

$$K_2\left[\cos\phi\frac{\partial^2\phi}{\partial z^2} - \left(\frac{\partial\phi}{\partial z}\right)^2 \sin\phi\right]\cos\phi + K_3\left[\sin\phi\frac{\partial^2\phi}{\partial z^2} + \left(\frac{\partial\phi}{\partial z}\right)^2\cos\phi\right]\sin\phi - 2\lambda_5\frac{\partial\phi}{\partial t} - (\tau_1+\tau_5)\frac{\partial u}{\partial z}\cos\phi - (\lambda_5+\lambda_2\cos 2\phi)\frac{\partial v}{\partial z} = 0 \tag{68}$$

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while the equations for linear momentum yield

$$\rho\frac{\partial u}{\partial t}=\frac{\partial}{\partial z}\left[\xi_1(\phi)\frac{\partial u}{\partial z}+\xi_2(\phi)\frac{\partial v}{\partial z}\cos\phi + (\tau_1+\tau_5)\cos\phi\frac{\partial\phi}{\partial t}\right]$$
$$\rho\frac{\partial v}{\partial t}=\frac{\partial}{\partial z}\left[\xi_2(\phi)\frac{\partial u}{\partial z}\cos\phi+\xi_3(\phi)\frac{\partial v}{\partial z} + (\lambda_5+\lambda_2\cos 2\phi)\frac{\partial\phi}{\partial t}\right] \quad (69)$$

$$\begin{aligned}2\xi_1(\phi)&=\mu_0+\mu_2+2\lambda_1+\lambda_4 + (\mu_4+\mu_5-2\lambda_2+2\lambda_3+\lambda_5+\lambda_6)\sin^2\phi\\ 2\xi_2(\phi)&=\kappa_1+\tau_1+\tau_2+\tau_5+2(\kappa_3+\tau_4)\sin^2\phi\\ 2\xi_3(\phi)&=\mu_0+\mu_4+\lambda_5+2\mu_3\sin^2\phi\cos^2\phi+2\lambda_2\cos 2\phi.\end{aligned} \quad (70)$$

At the plates one has

$$\phi(\pm d,t)=u(\pm d,t)=v(\pm d,t)=0, \qquad t\geqslant 0 \quad (71)$$

while the initial conditions are

$$\phi(z,0)=\phi_0(z), \qquad u(z,0)=v(z,0)=0, \qquad -d<z<d \quad (72)$$

$\phi_0(z)$ is again a known even function of z. This system of equations is almost identical to the system obtained in the previous section, differing only in the nature of the elastic terms in angular momentum and the particular forms of the coefficients. Clearly they invite similar approximate analyses as employed above.

Turning first to the half-space problem we again consider the application of a very strong field and therefore assume that

$$\phi_0(z)\simeq\frac{\pi}{2}. \quad (73)$$

Substitution of this value into the coefficients in equations (68) and (69) quickly yields

$$\begin{aligned}&K_3\frac{\partial^2\phi}{\partial z^2}-2\lambda_5\frac{\partial\phi}{\partial t}-(\lambda_5-\lambda_2)\frac{\partial v}{\partial z}=0\\ &\rho\frac{\partial u}{\partial t}=\xi_1\left(\frac{\pi}{2}\right)\frac{\partial^2 u}{\partial z^2}\\ &\rho\frac{\partial v}{\partial t}=\xi_3\left(\frac{\pi}{2}\right)\frac{\partial^2 v}{\partial z^2}+(\lambda_5-\lambda_2)\frac{\partial^2\phi}{\partial z\partial t}\end{aligned} \quad (74)$$

and selecting the origin at the plate these are subject to

$$\phi(0,t)=0, \qquad u(0,t)=v(0,t)=0, \qquad t\geqslant 0 \quad (75)$$

and

$$\phi(z,0)=\frac{\pi}{2}, \qquad u(z,0)=v(z,0)=0, \qquad z>0. \quad (76)$$

Here one quickly concludes that the component u is identically zero, and by comparison with the results of §3.1 that

$$\begin{aligned}&\phi\simeq\frac{\pi}{2}\,\mathrm{erf}(\kappa^{1/2}\zeta)\\ &v\simeq\frac{(\lambda_2-\lambda_5)}{\xi_3}\left(\frac{\pi k}{2\kappa t}\right)^{1/2}\left[\exp(-\kappa\zeta^2)-\exp\left(-\frac{\varepsilon\zeta^2}{\kappa}\right)\right]\end{aligned} \quad (77)$$

where

$$\zeta=\frac{z}{2(kt)^{1/2}}, \qquad k=\frac{K_3}{2\lambda_5}, \qquad \varepsilon=\frac{\rho K_3}{\xi_3\lambda_5}$$
$$\kappa=1-\frac{(\lambda_5-\lambda_2)^2}{2\lambda_5\xi_3}, \qquad 2\xi_3=\mu_0+\mu_4-2\lambda_2+\lambda_5. \quad (78)$$

From the inequality (21) one readily concludes that κ and ξ_3 are positive, and therefore ε is also positive. Again it is the solution for the flow component which is of interest. The solution takes the same form as in equations (49), but here it is not possible to say with any degree of certainty whether the combination $\lambda_2-\lambda_5$ is positive or negative. If the latter holds true, one obtains a similar graph for v to that depicted in figure 2 for u, and in this case one anticipates that the kickback effect is likely. However, if $\lambda_2-\lambda_5$ is positive, the induced backflow may simply hasten the relaxation process, with the **c**-director returning directly and more rapidly to the uniform configuration.

Finally we note that if the elastic constants K_2 and K_3 are equal one can set the value of ϕ in the coefficients of equations (68) and (69) equal to some fixed value to obtain

$$\left.\begin{aligned}2\lambda_5\frac{\partial\phi}{\partial t}&=K_3\frac{\partial^2\phi}{\partial z^2}-a\frac{\partial u}{\partial z}-b\frac{\partial v}{\partial z}\\ \rho\frac{\partial u}{\partial t}&=m\frac{\partial^2 u}{\partial z^2}+p\frac{\partial^2 v}{\partial z^2}+a\frac{\partial^2\phi}{\partial z\partial t}\\ \rho\frac{\partial v}{\partial t}&=n\frac{\partial^2 v}{\partial z^2}+p\frac{\partial^2 u}{\partial z^2}+b\frac{\partial^2\phi}{\partial z\partial t}\end{aligned}\right\} \quad (79)$$

where for the present problem

$$\begin{aligned}&a=(\tau_1+\tau_5)\cos\phi, \qquad b=\lambda_5+\lambda_2\cos 2\phi\\ &m=\xi_1(\phi), \qquad p=\xi_2(\phi)\cos\phi, \qquad n=\xi_3(\phi)\end{aligned} \quad (80)$$

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ϕ is the constant value assigned to ϕ in the coefficients. Clearly the above system is identical to the equations (51) allowing a repetition of the analysis of §3.2. However, we do not pursue this here, essentially because of lack of information about the various viscous coefficients that appear in the coefficients listed in equations (80).

5. Conclusions

The above solutions illustrate that the relaxation process of the **c**-director of a smectic C liquid crystal on the removal of an applied field is not entirely straightforward, it being necessary to take backflow into account. The analysis presented shows that in the case of homeotropic alignment, choosing reasonable values for the various material parameters, **c** initially moves in a direction contrary to that which one would expect. This phenomenon is due to the presence of backflow, although it is somewhat unclear precisely how the backflow influences the relaxation, due to the presence of two velocity components. However, it seems reasonable to assume that soon after the field is removed, the x-component of flow is the more influential, our analysis of relaxation at a single boundary confirming this. In this case the sign of u is important and the various parameters involved indicate that for most materials the flow is in the positive x-direction, coincident with the initial direction of the **c**-director. For a cell with a second boundary present the same argument suggests that a shear flow is induced in the cell resulting in a 'kickback' effect. Moreover, the calculation given for the relaxation in a cell amply confirms this. It is interesting to note that if $\tau_5 - \tau_1$ is negative, the induced flow assists the relaxation, rather than initially combatting it.

One weakness of our analysis, however, is that it assumes strong anchoring of the **c**-director at the boundaries, which may not prove to be such a reasonable assumption. With weak anchoring one would still expect backflow to have some influence, but perhaps not so pronounced. In the bookshelf geometry this aspect does not cause the same concern as it is more reasonable to assume strong anchoring. However, for this geometry the problem is more complex, not because of the governing equations, which are intrinsically the same as those for homeotropic alignment, but because of the number of material parameters involved, for which we have at present no values available. However, our preliminary analysis does indicate that backflow may influence the relaxation of the **c**-director in a non-monotonic manner.

The work reported in this paper was supported by the SERC and DRA Malvern through a CASE studentship for G.I.B.

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